Triangular Trigonometric Patches for Surface Interpolant

Xiang Fang & Stephen Mann
University of Waterloo

Cheriton School of Computer Science
University of Waterloo
200 University Avenue West
Waterloo, Ontario, Canada N2L 3G1

Cubic Triangular Bézier Patch

A cubic triangular Bézier patch is defined as

$$b^n(\boldsymbol{u}) = \sum_{|\boldsymbol{i}|=n} p_{\boldsymbol{i}} B^n_{\boldsymbol{i}}(\boldsymbol{u}),$$

where $p_{\boldsymbol{i}}$ are control points, $B^n_{\boldsymbol{i}}(\boldsymbol{u})$ are the *bivariate Bernstein polynomials* (blending functions),

$$B^n_{\boldsymbol{i}}(\boldsymbol{u}) = \binom{n}{\boldsymbol{i}} u^i v^j w^k,$$

with $\boldsymbol{u} = (u, v, w)$ being barycentric coordinates relative to a domain triangle, $\boldsymbol{i} = (i, j, k)$ is a multi-index with $n = |\boldsymbol{i}| = i + j + k$, where $\binom{n}{\boldsymbol{i}} = \frac{n!}{i!j!k!}$.

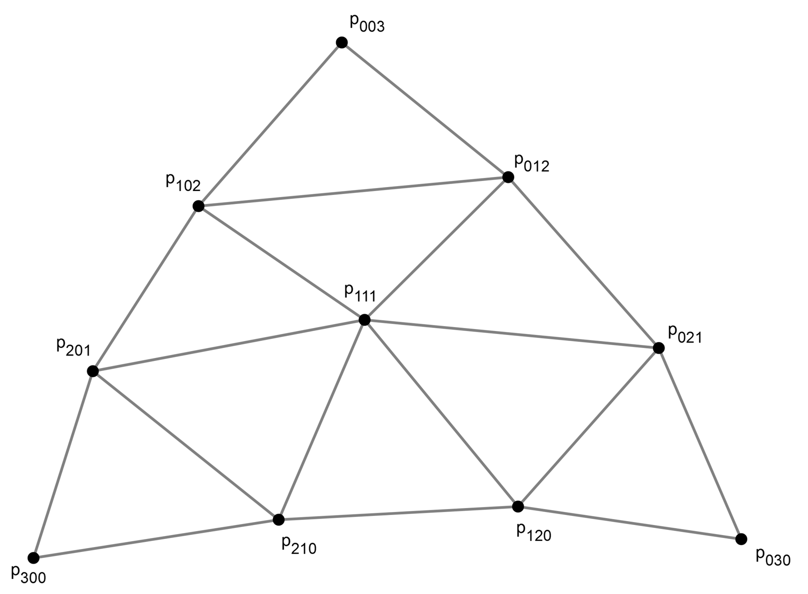

Figure 1: Layout of a cubic triangular Bézier / trigonometric patch.

Cubic Triangular Trigonometric Patch

A cubic triangular trigonometric patch uses the same layout of control points. Its blending functions are

$$\begin{aligned} f_{300}(\boldsymbol{u}) &= (1-d)^2, & f_{102}(\boldsymbol{u}) &= 2ae(1-f), \\ f_{210}(\boldsymbol{u}) &= 2b(1-d)f, & f_{030}(\boldsymbol{u}) &= (1-e)^2, \\ f_{201}(\boldsymbol{u}) &= 2c(1-d)e, & f_{021}(\boldsymbol{u}) &= 2cd(1-e), \\ f_{120}(\boldsymbol{u}) &= 2a(1-e)f, & f_{012}(\boldsymbol{u}) &= 2bd(1-f), \\ f_{111}(\boldsymbol{u}) &= 2abc, & f_{003}(\boldsymbol{u}) &= (1-f)^2, \end{aligned}$$

where

$$\begin{aligned} a &= \sin\tfrac{\pi u}{2}, & d &= \sin\tfrac{\pi(v+w)}{2}, \\ b &= \sin\tfrac{\pi v}{2}, & e &= \sin\tfrac{\pi(w+u)}{2}, \\ c &= \sin\tfrac{\pi w}{2}, & f &= \sin\tfrac{\pi(u+v)}{2}. \end{aligned}$$

Cubic triangular trigonometric (CTT) patches have the same C^1 continuity conditions as cubic triangular Bézier patches. The cubic triangular Bézier patches in any C^1 interpolant scheme can be replaced by the CTT patches, with the resulting CTT patches meeting C^1. For example, Clough-Tocher's construction [1] can use CTT patches instead of cubic Bézier patches.

Divided Trigonometric Patch

The center control point $p_{111} <$ of the cubic triangular trigonometric patch may be divided into four:

$$\begin{aligned} f_{111,0}(\boldsymbol{u}) &= 2abc(\tfrac{b^2c^2}{e^2} + \tfrac{b^2c^2}{f^2}), \\ f_{111,1}(\boldsymbol{u}) &= 2abc(\tfrac{c^2a^2}{f^2} + \tfrac{c^2a^2}{d^2}), \\ f_{111,2}(\boldsymbol{u}) &= 2abc(\tfrac{a^2b^2}{d^2} + \tfrac{a^2b^2}{e^2}), \\ f_{111}(\boldsymbol{u}) &= 2abc - f_{111,0}(\boldsymbol{u}) - f_{111,1}(\boldsymbol{u}) - f_{111,2}(\boldsymbol{u}). \end{aligned}$$

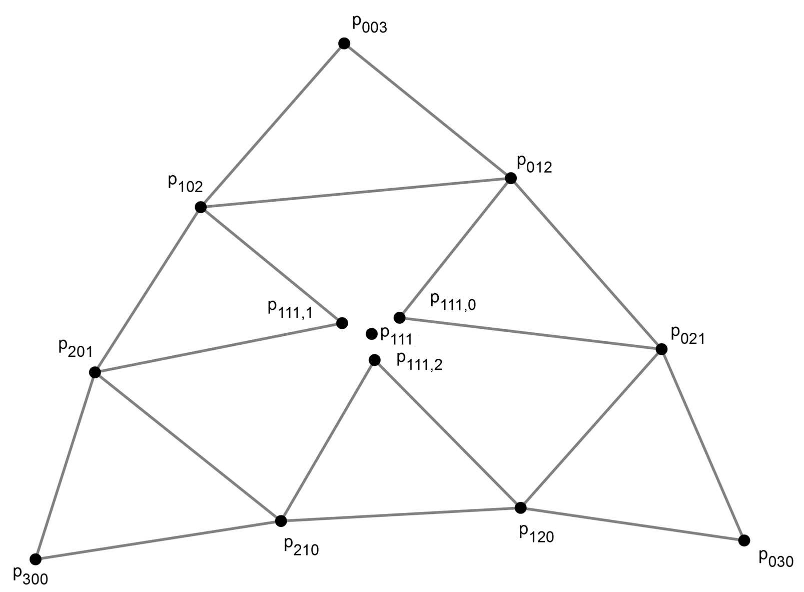

Figure 2: Layout of a divided trigonometric patch.

With the divided trigonometric patch, three of the four blended center points are related to one boundary each. Each of these three center points affects only the C^1 continuity condition of one boundary and does not affect the continuity across the other two boundaries.

The fourth center point is a "free" control point, and does not affect the continuity conditions across any boundary.

Bibliography

[1] Clough, Tocher, Finite element stiffness matrices for analysis of plate bending, Write-Patterson I, 1965.

Examples

Adjusting the fourth center point

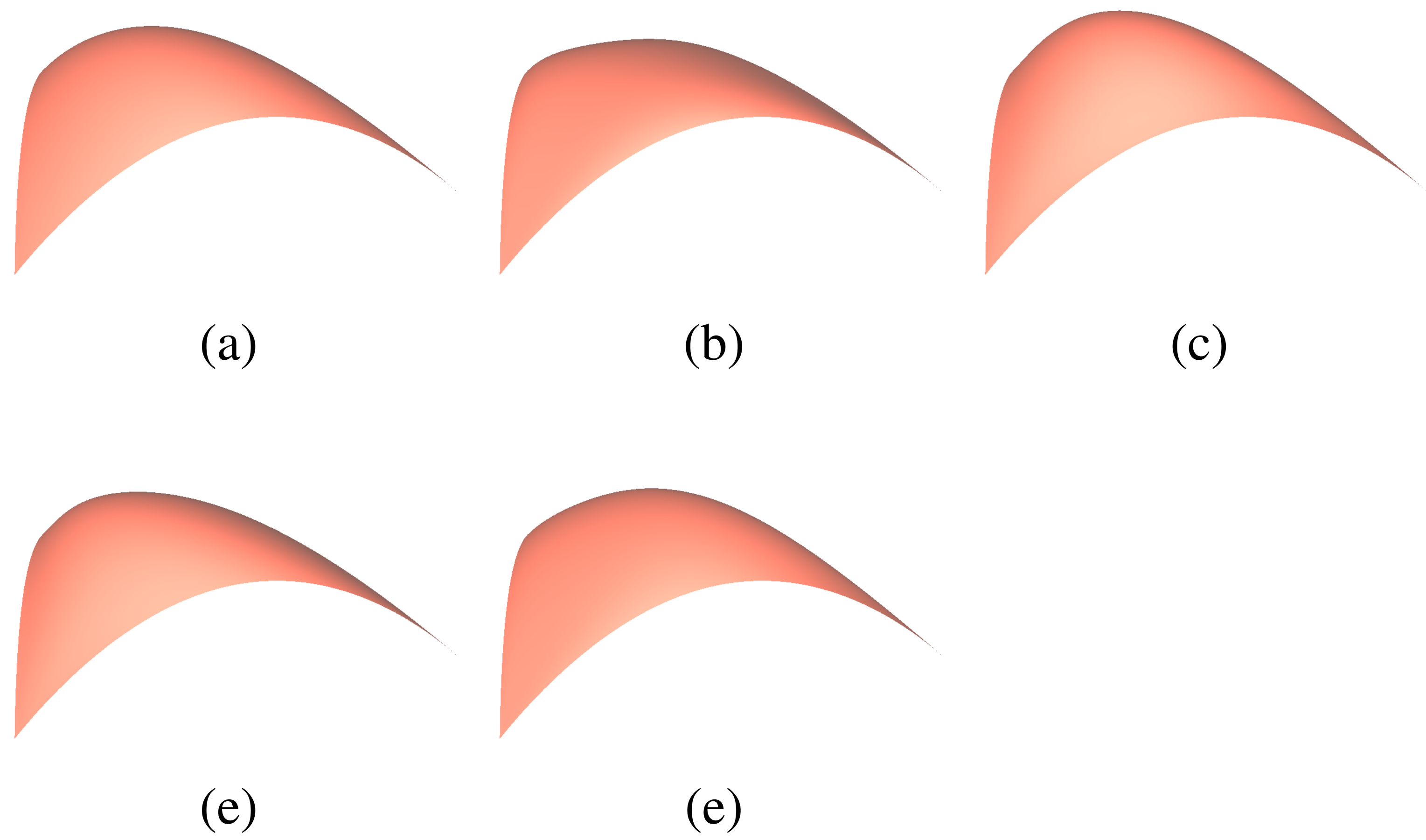

Figure 3: (a) The original patch surface. (b) Shift the center point downward. (c) Shift the center point upward. (d) Shift the center point left. (e) Shift the center point right.

Simple data fitting scheme

A simple data fitting scheme to test the divided trigonometric patch.

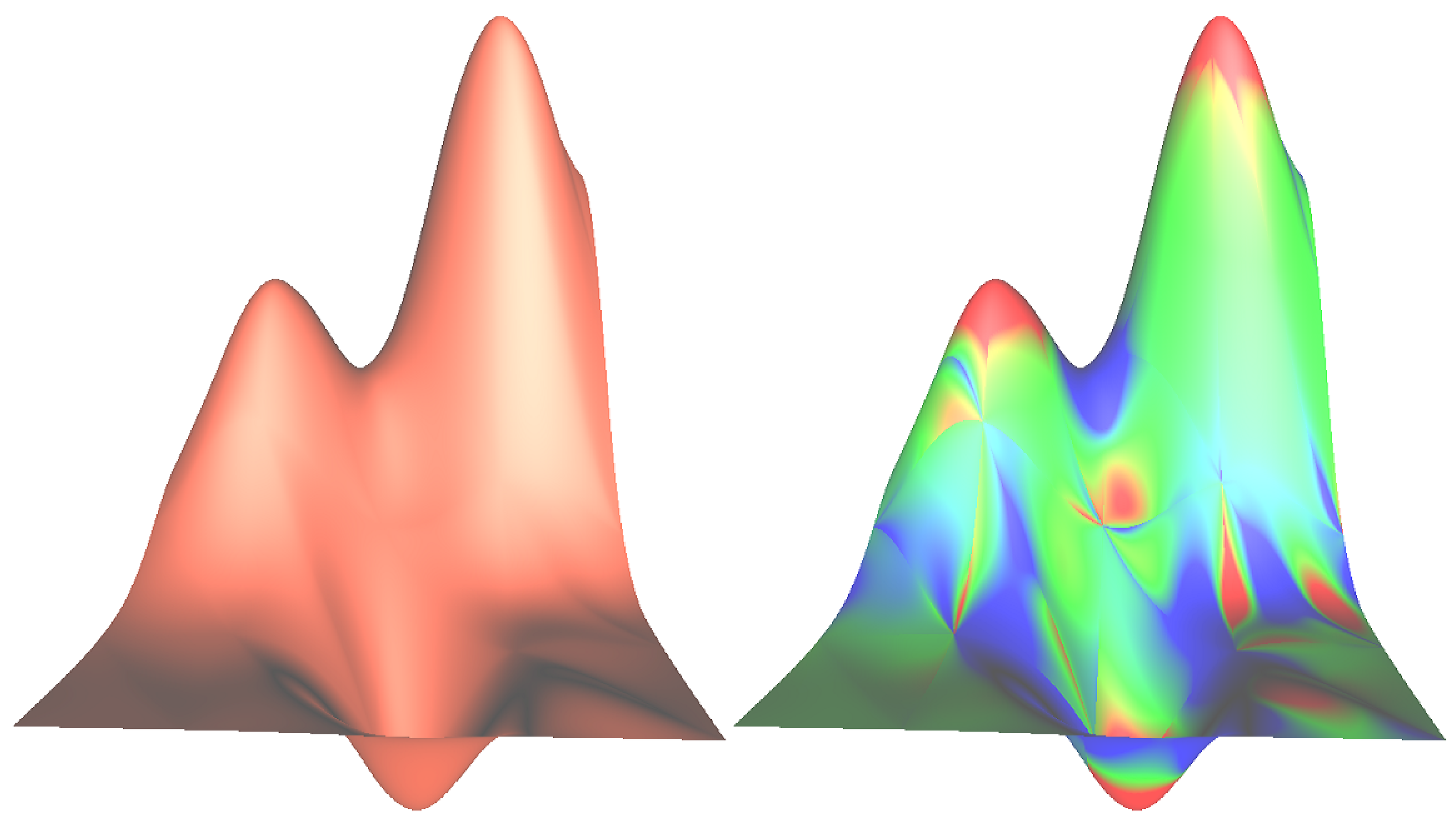

Figure 4: Trigonometric surface and its Gaussian curvature plot (curvature computed numerically).