Tessellating Algebraic Curves and Surfaces Using A-Patches

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Abstract

An implicit surface is defined as the zero set of a scalar function over 3-space. The sign of the implicit function determines whether a point is inside or outside the implicit surface:

$$f(P)\begin{cases} = 0 & P \text{ is on the surface.} \\ < 0 & P \text{ is inside the surface.} \\ > 0 & P \text{ is outside the surface.} \end{cases}$$

An algebraic function is an implicit function where the function is polynomial. By using an A-patch representation of algebraic curves and surfaces, we are able to identify regions in which the curve/surface does not lie, and efficiently tessellate the curve/surface in the regions in which it does lie.

A-patches

Trivariate Bernstein polynomials:

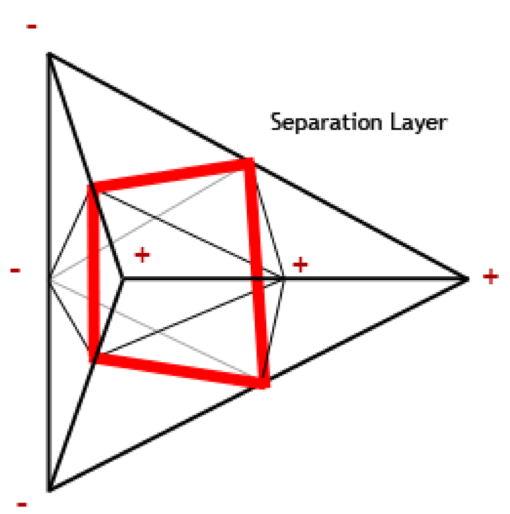
$$B^n_{\vec{i}}(P) = \binom{n}{\vec{i}} p_0^{i_0} p_1^{i_1} p_2^{i_2} p_3^{i_3},$$

where $\vec{i} = (i_0, i_1, i_2, i_3)$ with $i_0, i_1, i_2, i_3 \leq n$ and $i_0 + i_1 + i_2 + i_3 = n$, and where (p_0, p_1, p_2, p_3) are the Barycentric coordinates.

An A-patch weighs scalar values with the Bernstein basis:

$$F(P) = \sum_{\vec{i}} c_i B^n_{\vec{i}}(P).$$

A-patch coefficients are separated by sign with an intermediate layer of mixed signs:

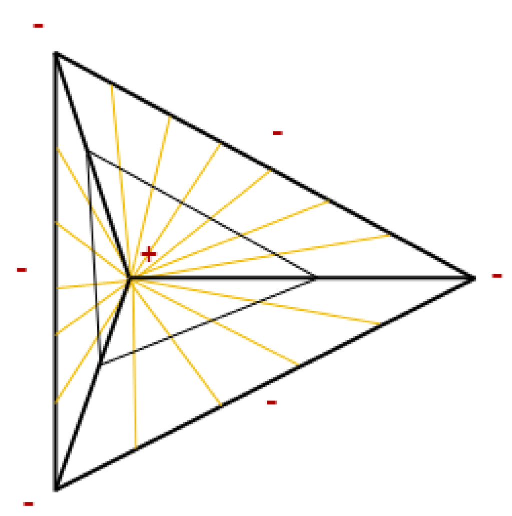


A Separation Layer for a Quadratic A-Patch

Advantages of A-patches

- Separation guarantees that a single sheet of the algebraic surface passes through the tetrahedron.
- Single sheet give means of tessellating:

  Root find along lines from corner to opposite face (if 3-sided A-Patch) or edge to opposite edge (if 4-sided A-Patch)

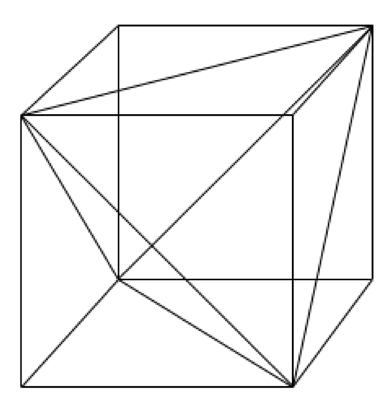


Plotting the Surface from an A-Patch

Surfaces

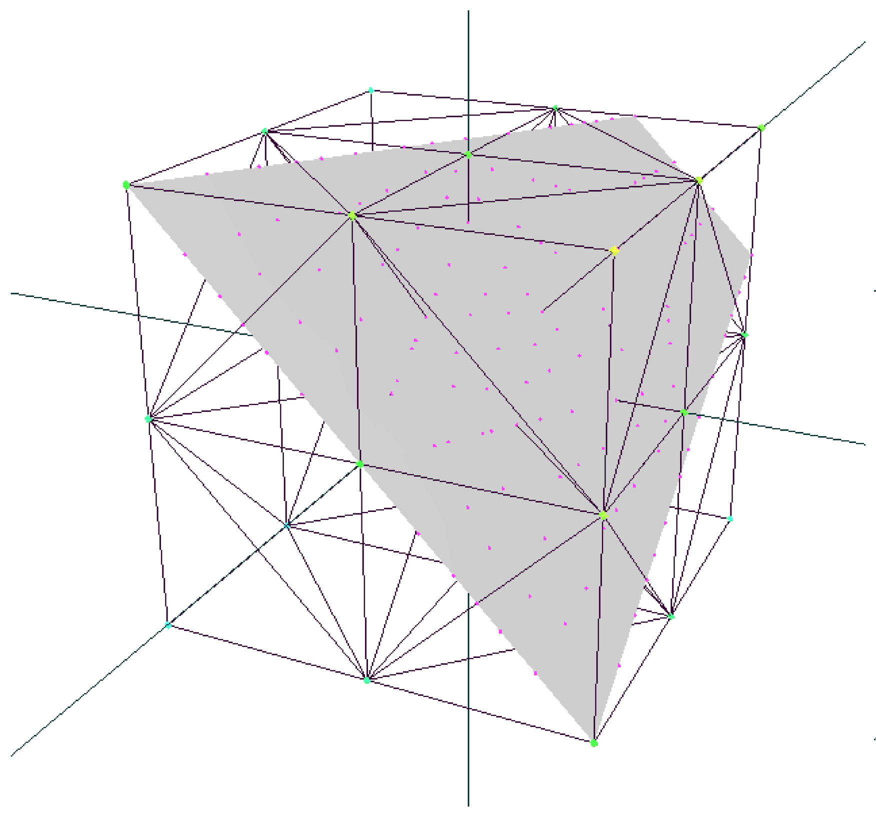
These observations lead to the following algorithm for tessellating algebraic surfaces:

- Tetrahedralize space into octets of cubic units of A-Patches (5 in each cube)
- Bernstein form of algebraic within each cubic unit
- For each cubic unit, test the coefficients:
  - All patches have all positive/negative coefficients: surface does not pass through the area
  - All patches are in A-patch format: tessellate
  - Any patch has mixed coefficients: subdivide cubic unit into new octet

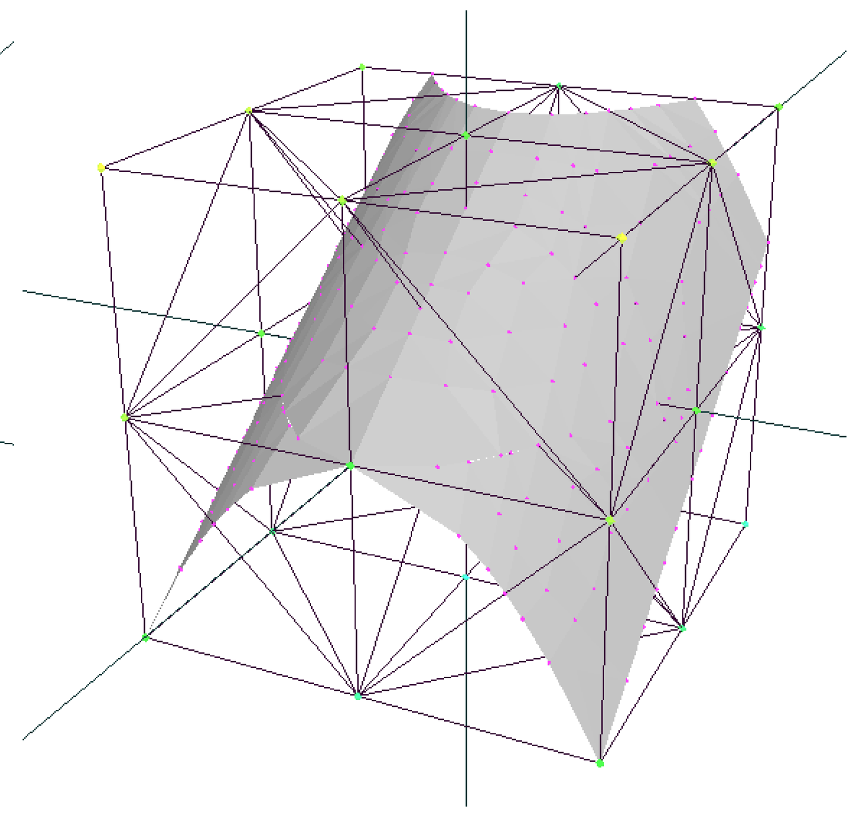


A Representation of a Cubic Unit of A-Patches

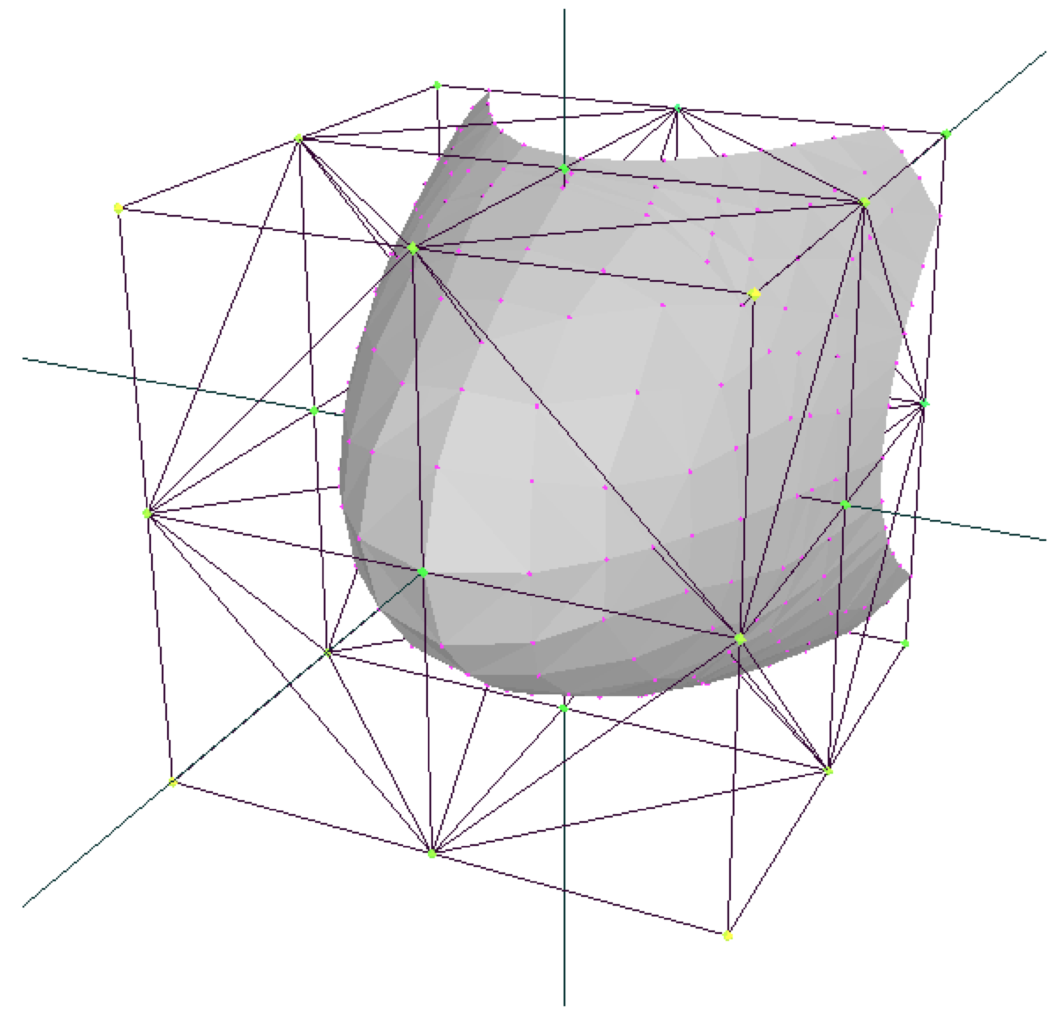
Examples



$x + y + 0.7z = 0$



$x^2 + y + 0.7z = 0$



$x^2 + 0.8y^2 + 0.7z = 0$

Curves

The algorithm for tessellating algebraic curves:

- Make an initial triangular "grid" over the region of interest.
- For each triangle,
  - Convert the algebraic function to Bernstein-Bézier representation for that triangle.
  - Test the coefficients:
    - If all of same sign: curve/surface does not pass through
    - If in A-patch format: tessellate
    - Otherwise: subdivide the triangle and repeat the process.