

G^2 Pythagorean Hodograph Curves

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Abstract

I rederive the cubic G^1 Hermite interpolatory Pythagorean Hodograph curve construction of Meek and Walton [MeekWalton97]. I then use the new approach to rederive the quartic Pythagorean Hodograph curve construction of Wang and Fang [WangFang09], and show how to use this curve to get G^2 continuity between adjacent quartic segments. Finally, I derive a quintic G^2 Hermite interpolatory Pythagorean Hodograph curve construction.

Pythagorean Hodograph Curves

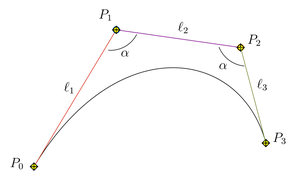
A curve B is a Pythagorean-hodograph (PH) curve [Farouki] if

$$||B'(t)|| = \sigma(t)$$

for some polynomial $\sigma(t)$.

Cubic PH curve $B(t) = \sum_{i=0}^{3} P_i B_i^3(t)$:

- Interior angle are equal
- Edge lengths ℓ_1, ℓ_2, ℓ_3 in geometric progression ($\ell_1\ell_3 = \ell_2^2$)



Cubic PH curve

Cubic G^1 Hermite

Given: two points P_1 and P_2 and two vector $\vec{v}_0$ and $\vec{v}_1$,

Find control points for PH $C(t) = \sum_{i=0}^{3} P_i B_i^3(t)$ such that $C(0) = P_1$, $C(1) = P_2$ and $C'(0) = \alpha_1\vec{v}_1$ and $C'(1) = \alpha_2\vec{v}_2$ for some scalars $\alpha_1, \alpha_2 > 0$

Meek and Walton solved for cubic in complex plane

References

[Farouki] R. T. Farouki, *Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable*, Springer Berlin, Heidelberg, 2008. https://doi.org/10.1007/978-3-540-73398-0

[HRV24] Hormann, K., Romani, L., and Viscardi, A., *New algebraic and geometric characterizations of planar quintic Pythagorean-hodograph curves*, Journal of Computer Aided Geometric Design, 108 (2024), pg 1–14.

[Mann26] Mann, S, *Curvature Continuous Piecewise Pythagorean Hodograph Curves*, Research Report CS-2026-01, David R Cheriton School of Computer Science, University of Waterloo (February 2026)

[MeekWalton97] Meek, D. S. and Walton, D. J., *Geometric Hermite interpolation with Tschirnhausen cubits*, Journal of Computational and Applied Mathematics, 81 (1997) pg 299–309

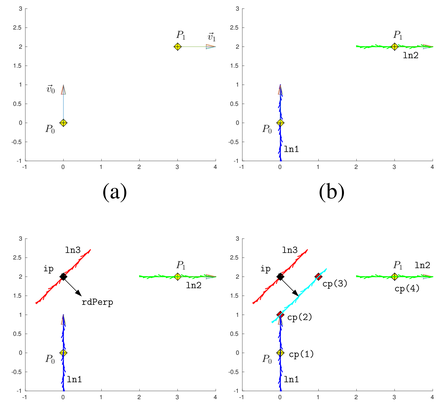
[WangFang09] Wang, G. and Fang, L., *On control polygons of quartic Pythagorean¡E2¿¡80¿¡93¿hodograph curves*, Journal of Computer Aided Geometric Design, 26 (2009) pg 1006–1015.

Alternate approach in real plane

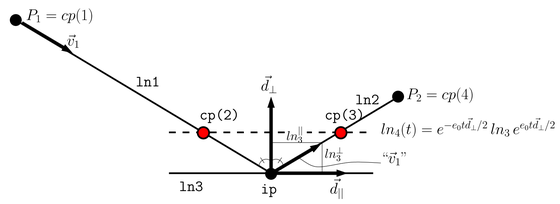
(b) Construct the two lines, `ln1` and `ln2`, from P_i, $\vec{v}_i$

(c) Construct "halfway line" `ln3` thru intersection point `ip`

(d) Derive a quadratic equation to translate the interior edge of the Bézier control polygon (`ln3`) so that the edge length ratio constraint of a Pythagorean hodograph curve is met.



To derive quadratic: Set up local coordinate system at intersection point,



As we vary t, ℓ_1, ℓ_2, and ℓ_3 will vary:

$$\ell_1 = l_1 - \frac{t}{ln_3^\perp}, \qquad \ell_2 = 2t/s, \qquad \ell_3 = l_2 - \frac{t}{ln_3^\perp}$$

Expanding the ratio condition $\ell_1\ell_3 = \ell_2^2$ gives us

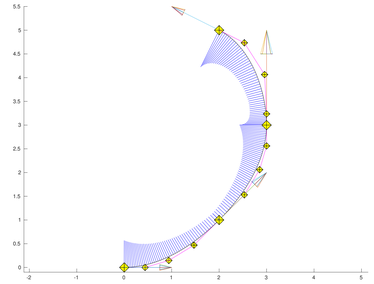
$$(\ell_1 - \frac{t}{ln_3^\perp})(\ell_2 - \frac{t}{ln_3^\perp}) = \left(\frac{2}{s}t\right)^2$$

$$\Rightarrow \quad \ell_1\ell_2 - (\ell_1 + \ell_2)\frac{t}{ln_3^\perp} + \left(\frac{1}{(ln_3^\perp)^2} - \frac{4}{s^2}\right)t^2 = 0.$$

Similar approach works for quartic; however, one degree of freedom left over. Use extra degree of freedom to interpolate curvature on one end

Result is seven equations in seven unknowns; reduces to a quartic equation

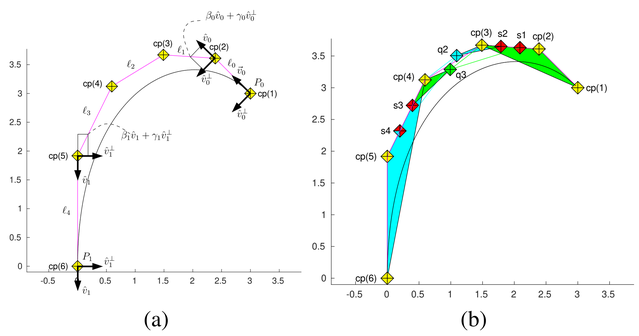
Quartic Example



Three quartic PH curve segments meeting G^2

Quintic PH Curves

Hormann et al. [HRV24] similar triangle conditions



Trisect the segment `cp(2),cp(3)` (giving `s1` and `s2`) and trisect the segment `cp(4),cp(5).` (giving `s3` and `s4`). Then construct `q3` and `q2` such that we have two sets of similar triangles:

$$\triangle s_1, cp(3), q_3 \sim \triangle cp(1), cp(2), cp(3)$$
$$\triangle q_2, cp(4), s_4 \sim \triangle cp(4), cp(5), cp(6).$$

Then Hormann et al. showed that the curve is a PH curve if and only if

$$\triangle q_3, cp(4), s_3 \sim \triangle cp(1), cp(2), cp(3)$$
$$\triangle s_2, cp(3), q_2 \sim \triangle cp(4), cp(5), cp(6),$$

with the additional requirement that $\ell_1, \ell_3 \neq 0$.

Quintic G^2 PH Construction

Start with C^0 and G^1 conditions on control points

Convert similar triangle conditions to conditions on control points

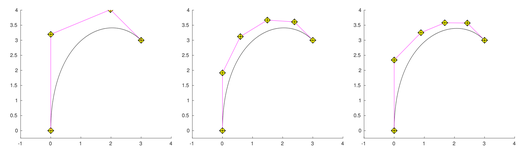
Add G^2 constraints

Eight non-linear equations, eight unknowns.

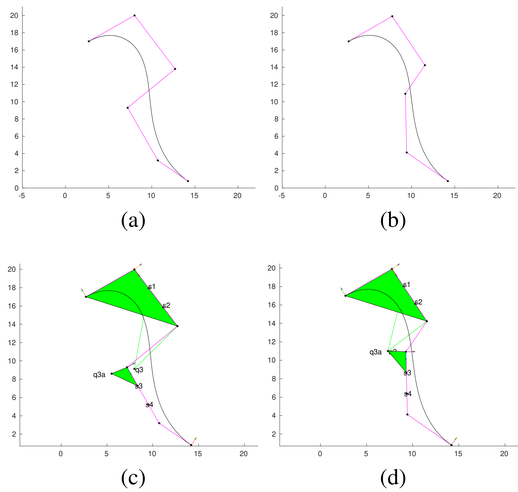
Solve Numerically

Initial conditions:

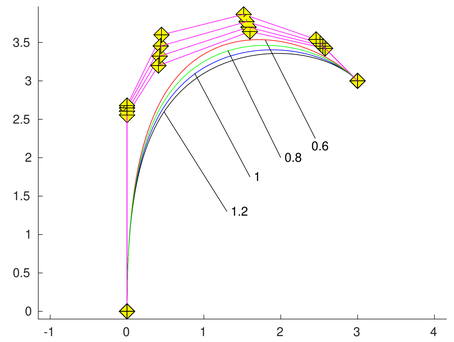
$$\ell_i = |P_1 - P_0|/5, \quad \beta_0 = \beta_1 = 1, \quad \gamma_0 = \pm 1, \quad \gamma_1 = \pm 1.$$



(a) Cubic PH curve; (b) Quintic PH curve using initial conditions from cubic; (c) Quintic PH curve using above initial conditions



(a) Cubic curve; (b) Quintic PH fit to position, tangent, curvature of cubic; (c) Similar triangle condition fails for cubic; (d) similar triangle condition works for quintic PH.



$P_0 = (0,0)$, $\vec{v}_0 = e2$, $\kappa_0 = 0.05$;
$P_1 = (3,3)$, $\vec{v}_1 = \sqrt{(2)}/2(e_1 + -e_2)$. Red: $\kappa_1 = 0.6$. Green: $\kappa_1 = 0.8$. Blue: $\kappa_1 = 1$. Black: $\kappa_1 = 1.2$